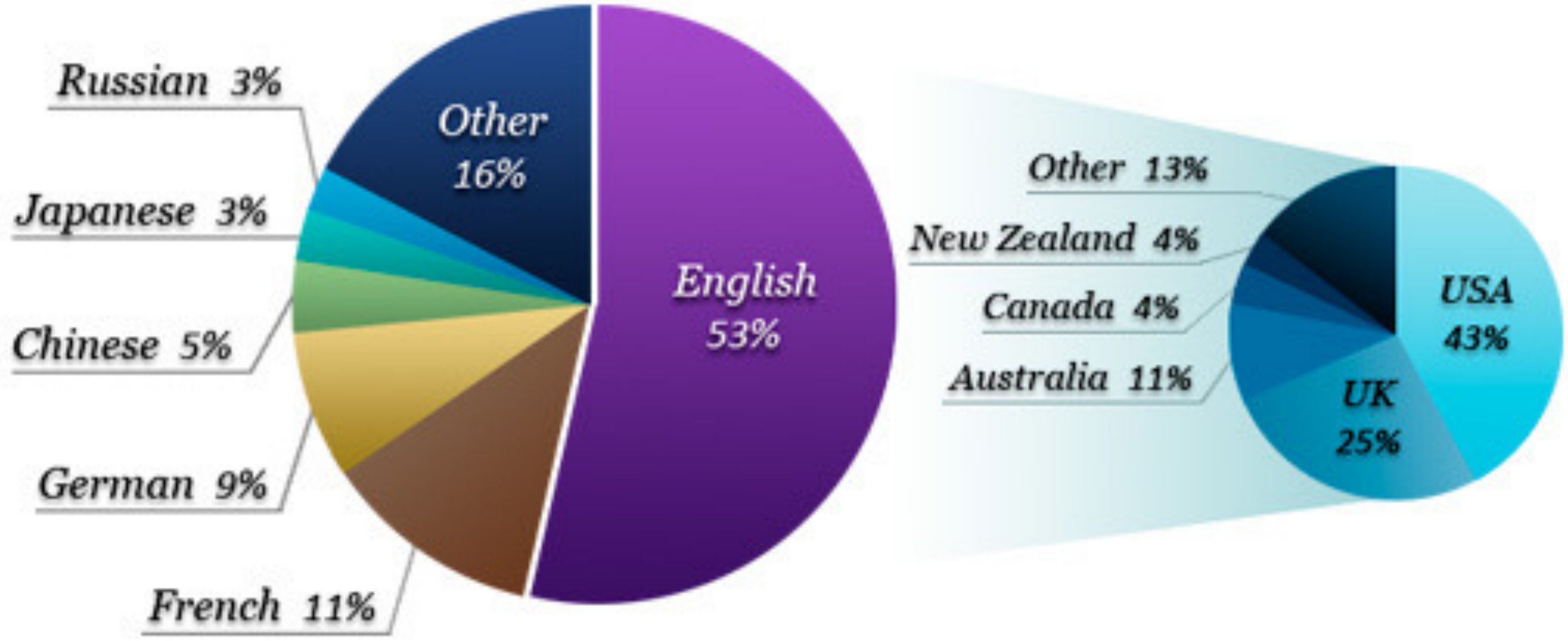
Introduction. Operations on matrices

Dominance of English Language



Source: British Council, 2006

https://www.youtube.com/watch?v=egi3bPp3CCM

Matrices

$$A = \begin{bmatrix} 3 & 1 & -6 \\ 2 & 1 & 9 \end{bmatrix}$$

What is the order of this matrix?

Matrices

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Order is 2 rows and 3 columns (RxC) = 2x3.

But then what is value of $A_{23} =?$

Matrices

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Order is 2 rows and 3 columns (RxC) = 2x3.

Value of $A_{23} = 9$

Coffee

- Suppose that you are responsible for selling coffee for your company's staff. The weekly sales rates for the five different types and sizes of coffee that are available are: Standard: 139 tenge, Intermediate: 160 tenge, Large: 205 tenge, Latte: 340 tenge and Luxury Latte: 430 tenge. For next week you know that your coffee requirements will be: 4 Standard, 3 Intermediate, 12 Large, 2 Latte and 1 Luxury Latte. How would you work out the total bill?

Coffee

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- $4 \times 139 + 3 \times 160 + 12 \times 205 + 2 \times 340 + 1 \times 430 = 4,606$

Coffee

- If you know that your coffee requirements will change from week to week, it can help make calculations clearer if the number of coffees required in each category are set out in tabular form.

Type	Week 1	Week 2	Week 3
Standard	4	7	2
Intermediate	3	5	5
Large	12	9	5
Latte	2	1	3
Luxury Latte	1	1	2

Coffee

- The total coffee bill for each week can then be calculated by multiplying the number of coffees in each category by the corresponding price.
- A matrix is defined as an array of numbers (or algebraic symbols) set out in rows and columns. Therefore, the coffee requirements for the 3-week period in this example can be set out as the matrix

$$A = \begin{bmatrix} 4 & 7 & 2 \\ 3 & 5 & 5 \\ 12 & 9 & 5 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

- where each row corresponds to a type of coffee and each column corresponds to a week. The usual notation system is to denote matrices by a capital letter in bold type, as for matrix A above, and to enclose the elements of a matrix in a set of squared brackets, i.e. [ ].

Vectors

- Matrices with only one column or row are known as vectors. These are usually represented by lower case letters, in bold. For example, the set of coffee prices we started this presentation with can be specified (in tenge) as the 1 × 5 row vector

$$p = [139 \quad 160 \quad 205 \quad 340 \quad 430]$$

- and the coffee requirements in week 1 can be specified as the 5 × 1 column vector

$$q = \begin{bmatrix} 4 \\ 3 \\ 12 \\ 2 \\ 1 \end{bmatrix}$$

Matrix addition and subtraction

- Matrices that have the same order can be added together, or subtracted. The addition, or subtraction, is performed on each of the corresponding elements.

Example

- A retailer sells two products, Toys and Laptops, in two shops A and B. The number of items sold for the last 4 weeks in each shop are shown in the two matrices A and B below, where the columns represent weeks and the rows correspond to products Toys and Laptops, respectively.

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 12 & 7 \\ 10 & 12 & 9 & 14 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 8 & 9 & 3 & 4 \\ 8 & 18 & 21 & 5 \end{bmatrix}$$

- Derive a matrix for total sales for this retailer for these two products over the last 4 weeks.

Solution

- Total sales for each week will simply be the sum of the corresponding elements in matrices A and B. For example, in week 1 the total sales of product Toys will be 5 plus 8. Total combined sales for Toys and Laptops can therefore be represented by the matrix

$$\mathbf{T} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} 5 & 4 & 12 & 7 \\ 10 & 12 & 9 & 14 \end{bmatrix} + \begin{bmatrix} 8 & 9 & 3 & 4 \\ 8 & 18 & 21 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5+8 & 4+9 & 12+3 & 7+4 \\ 10+8 & 12+18 & 9+21 & 14+5 \end{bmatrix} = \begin{bmatrix} 13 & 13 & 15 & 11 \\ 18 & 30 & 30 & 19 \end{bmatrix}$$

Class work

If $\mathbf{A} = \begin{bmatrix} 12 & 30 \\ 8 & 15 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 7 & 35 \\ 4 & 8 \end{bmatrix}$ what is $\mathbf{A} - \mathbf{B}$?

Solution

If $\mathbf{A} = \begin{bmatrix} 12 & 30 \\ 8 & 15 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 7 & 35 \\ 4 & 8 \end{bmatrix}$ what is $\mathbf{A} - \mathbf{B}$?

Solution

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 12 & 30 \\ 8 & 15 \end{bmatrix} - \begin{bmatrix} 7 & 35 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 12 - 7 & 30 - 35 \\ 8 - 4 & 15 - 8 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 4 & 7 \end{bmatrix}$$

Multiplication

- There are two forms of multiplication that can be performed on matrices. A matrix can be multiplied by a specific value, such as a number (scalar multiplication) or by another matrix (matrix multiplication). Scalar multiplication simply involves the multiplication of each element in a matrix by the scalar value, as in Example in slide #17-18. Matrix multiplication is rather more complex and is explained later, in slide #22

Scalar multiplication

- The number of units of a product sold by a retailer for the last 2 weeks are shown in matrix A below, where the columns represent weeks and the rows correspond to the two different shop units that sold them.

$$\mathbf{A} = \begin{bmatrix} 12 & 30 \\ 8 & 15 \end{bmatrix}$$

- If each item sells for $4, derive a matrix for total sales revenue for this retailer for these two shop units over this two-week period.

Solution

- Total revenue is calculated by multiplying each element in matrix of sales quantities A by the scalar value 4, the price that each unit is sold at. Thus total revenue can be represented (in $) by the matrix

$$\mathbf{R} = 4\mathbf{A} = \begin{bmatrix} 4 \times 12 & 4 \times 30 \\ 4 \times 8 & 4 \times 15 \end{bmatrix} = \begin{bmatrix} 48 & 120 \\ 32 & 60 \end{bmatrix}$$

- Scalar division works in the same way as scalar multiplication, but with each element divided by the relevant scalar value.

Example

- If the set of coffee prices in the vector

$p = [139 \quad 160 \quad 205 \quad 340 \quad 430]$ includes VAT (Value Added Tax) at 17.5% and your company can claim this tax back, what is the vector v of prices without this tax?

Solution

1. First of all we need to find the scalar value used to scale down the original vector element values. As the tax rate is 17.5% then the quoted prices will be 117.5% times the basic price. Therefore a quoted price divided by 1.175 will be the basic price and so the vector of prices (in tenge) without the tax will be.

$$\mathbf{v} = \left(\frac{1}{1.175}\right)\mathbf{p} = \left(\frac{1}{1.175}\right)\begin{bmatrix}139 & 160 & 205 & 340 & 430\end{bmatrix}$$

$$= \begin{bmatrix}\left(\frac{1}{1.175}\right)139 & \left(\frac{1}{1.175}\right)160 & \left(\frac{1}{1.175}\right)205 & \left(\frac{1}{1.175}\right)340 & \left(\frac{1}{1.175}\right)430\end{bmatrix}$$

$$= \begin{bmatrix}118.30 & 136.17 & 174.47 & 289.36 & 365.96\end{bmatrix}$$

Solution

2. Multiply the price with VAT by 17.5 and divide by 117.5. Let's try to calculate VAT, select VAT from 139: 139x17.5/117.5=20.702. VAT is 20,702. Amount without VAT = 139 - 20.702 = 118.297.

Matrix multiplication

- If one matrix is multiplied by another matrix, the basic rule is to multiply elements along the rows of the first matrix by the corresponding elements down the columns of the second matrix. The easiest way to understand how this operation works is to first work through some examples that only involve matrices with one row or column, i.e. vectors.

$$\mathbf{p} = \begin{bmatrix} 139 & 160 & 205 & 340 & 430 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 4 \\ 3 \\ 12 \\ 2 \\ 1 \end{bmatrix}$$

Matrix multiplication

- In terms of these two vectors, what we have done is multiply the first element in the row vector p by the first element in the column vector q. Then, going across the row, the second element of p is multiplied by the second element down the column of q. The same procedure is followed for the other elements until we get to the end of the row and the bottom of the column.

- $4 \times 139 + 3 \times 160 + 12 \times 205 + 2 \times 340 + 1 \times 430 = 4{,}606$

Matrix multiplication

- Now consider the situation where the coffee prices are still shown by the vector

$$p = [139 \quad 160 \quad 205 \quad 340 \quad 430]$$

- but there are now three weeks of different coffee requirements, shown by the columns of matrix

$$A = \begin{bmatrix} 4 & 7 & 2 \\ 3 & 5 & 5 \\ 12 & 9 & 5 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution

- To calculate the total coffee bill for each of the three weeks, we need to find the vector

T=pA

- This should have the order 1 × 3, as there will be one element (i.e. the bill) for each of the three weeks. The first element of t is the bill for the first week, which we have already found in the example above. The coffee bill for the second week is worked out using the same method, but this time the elements across the row vector p multiply the elements down the second column of matrix A, giving
- $139 \times 7 + 160 \times 5 + 205 \times 9 + 340 \times 1 + 430 \times 1 = 4{,}388$

Solution

- The third element is calculated in the same manner, but working down the third column of A. The result of this matrix multiplication exercise is therefore

$$\mathbf{t} = \mathbf{pA} = \begin{bmatrix} 139 & 160 & 205 & 340 & 430 \end{bmatrix} \begin{bmatrix} 4 & 7 & 2 \\ 3 & 5 & 5 \\ 12 & 9 & 5 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4606 & 4388 & 3983 \end{bmatrix}$$

Matrix multiplication

- The above examples have shown how the basic principle of matrix multiplication involves the elements across a row vector multiplying the elements down the columns of the matrix being multiplied, and then summing all the products obtained. If the first matrix has more than one row (i.e. it is not a vector) then the same procedure is followed across each row. This means that the number of rows in the final product matrix will correspond to the number of rows in the first matrix.

Example

Multiply the two matrices $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 8 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 7 & 5 & 2 \\ 4 & 8 & 1 \end{bmatrix}$

Solution

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 7 & 5 & 2 \\ 4 & 8 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 7 + 3 \times 4 & 2 \times 5 + 3 \times 8 & 2 \times 2 + 3 \times 1 \\ 8 \times 7 + 1 \times 4 & 8 \times 5 + 1 \times 8 & 8 \times 2 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 26 & 34 & 7 \\ 60 & 48 & 17 \end{bmatrix}$$

Matrix multiplication

- You now may be wondering what happens if the number of elements along the rows of the first matrix (or vector) does not equal the number of elements in the columns of the matrix that it is multiplying. The answer to this question is that it is not possible to multiply two matrices if the number of columns in the first matrix does not equal the number of rows in the second matrix. Therefore, if a matrix A has order $m \times n$ and another matrix B has order $r \times s$, then the multiplication AB can only be performed if $n = r$, in which case the resulting matrix $C = AB$ will have order $(m \times s)$.

Matrix multiplication

- This principle is illustrated in Example above. Matrix A has order 2 × 2 and matrix B has order 2 × 3 and so the product matrix AB has order 2 × 3. Some other examples of how the order of different matrices affects the order of the product matrix when they are multiplied are given in Table below.

A	**B**	*Order of product matrix* **AB**
5×3	3×2	5×2
1×8	8×1	1×1
3×5	2×4	Matrix multiplication not possible
3×4	4×3	3×3
4×3	4×3	Matrix multiplication not possible

Class work

Find the product matrix $\mathbf{C} = \mathbf{AB}$ when

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 12 \\ 6 & 0 & 20 \\ 1 & 8 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 10 & 0.5 & 1 & 7 \\ 6 & 3 & 8 & 2.5 \\ 4 & 4 & 2 & 0 \end{bmatrix}$$

Solution

$$c_{11} = 4 \times 10 + 2 \times 6 + 12 \times 4 = 40 + 12 + 48 = 100$$

$$c_{12} = 4 \times 0.5 + 2 \times 3 + 12 \times 4 = 2 + 6 + 48 = 56$$

$$c_{13} = 4 \times 1 + 2 \times 8 + 12 \times 2 = 4 + 16 + 24 = 44$$

$$c_{14} = 4 \times 7 + 2 \times 2.5 + 12 \times 0 = 28 + 5 + 0 = 33$$

$$c_{21} = 6 \times 10 + 0 \times 6 + 20 \times 4 = 60 + 0 + 80 = 140$$

$$c_{22} = 6 \times 0.5 + 0 \times 3 + 20 \times 4 = 3 + 0 + 80 = 83$$

$$c_{23} = 6 \times 1 + 0 \times 8 + 20 \times 2 = 6 + 0 + 40 = 46$$

$$c_{24} = 6 \times 7 + 0 \times 2.5 + 20 \times 0 = 42 + 0 + 0 = 42$$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} 100 & 56 & 44 & 33 \\ 140 & 83 & 46 & 42 \\ 78 & 44.5 & 75 & 27 \end{bmatrix}$$

inverse of a matrix

To actually find the inverse of a matrix, we first need to consider some special concepts associated with square matrices, namely:

- The Determinant
- Minors
- Cofactors
- The Adjoint Matrix

Determinants

- For a 2nd order matrix (i.e. order 2 × 2) the determinant is a number calculated by multiplying the elements in opposite corners and subtracting. The usual notation for a determinant is a set of vertical parallel lines either side of the array of elements, instead of the squared brackets used for a matrix. The determinant of the general 2 × 2 matrix A, written as |A|, will therefore be:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example

Find the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 5 & 7 \\ 4 & 9 \end{bmatrix}$

Solution

$$|\mathbf{A}| = \begin{vmatrix} 5 & 7 \\ 4 & 9 \end{vmatrix} = 5 \times 9 - 7 \times 4 = 45 - 28 = 17$$

The determinant of a 3rd order matrix

This entails multiplying each of the elements in the first row by the determinant of the matrix remaining when the corresponding row and column are deleted. For example, the element a11 is multiplied by the determinant of the matrix remaining when row 1 and column 1 are deleted from the original 3 × 3 matrix. If we start from a11 then, as we use this method for each element across the row, the sign of each term will be positive and negative alternately. Thus the second term has a negative sign.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|\mathbf{A}| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Derive the determinant of matrix $\mathbf{A} = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 5 & 2 \\ 9 & 0 & 4 \end{bmatrix}$

Solution

$$|\mathbf{A}| = 4\begin{vmatrix} 5 & 2 \\ 0 & 4 \end{vmatrix} - 6\begin{vmatrix} 2 & 2 \\ 9 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 9 & 0 \end{vmatrix}$$

$$= 4(20 - 0) - 6(8 - 18) + (0 - 45) = 80 + 60 - 45 = 95$$

Example

Derive the determinant of matrix $\mathbf{A} = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 5 & 2 \\ 9 & 0 & 4 \end{bmatrix}$ by expanding along the 3rd row.

Solution

$$|\mathbf{A}| = 9\begin{vmatrix} 6 & 1 \\ 5 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix}$$

$$= 9(12 - 5) - 0 + 4(20 - 12) = 63 + 32 = 95$$

Minors, cofactors and the Laplace expansion

- The Laplace expansion is a method that can be used to evaluate determinants of any order. Before explaining this method, we need to define a few more concepts (some of which we have actually already started using).

Minors

The minor $|\mathbf{M}_{ij}|$ of matrix $\mathbf{A}$ is the determinant of the matrix left when row i and column j have been deleted.

For example, if the first row and first column are deleted from matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant of the remaining matrix will be the minor

$$|\mathbf{M}_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Example

Find the minor $|\mathbf{M}_{31}|$ of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 2 & 3 \\ 1 & 9 & 4 \\ 4 & 3 & 6 \end{bmatrix}$$

Solution

$$|\mathbf{M}_{31}| = \begin{vmatrix} 2 & 3 \\ 9 & 4 \end{vmatrix} = 8 - 27 = -19$$

Cofactors

- A cofactor is the same as a minor, except that its sign is determined by the row and column that it corresponds to. The sign of cofactor |Cij | is equal to $(-1)^{i+j}$. Thus if the row number and column number sum to an odd number, the sign will be negative. For example, to derive the cofactor | C_{12} | for the general 3rd order matrix A we eliminate the 1st row and the 2nd column and then, since i + j = 3 , we multiply the determinant of the elements that remain by $(-1)^3$

$$|\mathbf{C}_{12}| = (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Example

Find the cofactor $|\mathbf{C}_{22}|$ of the matrix $\mathbf{A} = \begin{bmatrix} 8 & 2 & 3 \\ 1 & 9 & 4 \\ 4 & 3 & 6 \end{bmatrix}$

Solution

The cofactor $|\mathbf{C}_{22}|$ is the determinant of the matrix remaining when the 2nd row and 2nd column have been eliminated. It will have the sign $(-1)^4$ since $i + j = 4$. The solution is therefore

$$|\mathbf{C}_{22}| = (-1)^4 \begin{vmatrix} 8 & 3 \\ 4 & 6 \end{vmatrix} = (+1)(48 - 12) = 36$$

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